

# Exact Evaluation of the Accuracy of Diffusion Models for Inverse Problems with Gaussian Data Distributions



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## Objective

Compare algorithms based on diffusion models for solving inverse problems with Gaussian distributions.

## DDPM iterations for sampling $p_0(\cdot | v)$

$$v = Ax_0 + \sigma n, \quad x_0 \sim p_0, \sigma > 0, n \sim \mathcal{N}(0, I)$$

$$y_{t-1} = \frac{1}{\sqrt{\alpha_t}} (y_t + \beta_t \nabla \log p_t(y_t | v)) + \sqrt{\beta_t} z_t \quad \text{and by Bayes' rule, } \nabla_x \log p_t(x_t | v) = \underbrace{\nabla_x \log p_t(x_t)}_{\text{pretrained}} + \nabla_x \log p_t(v | x_t)$$

$$1 \leq t \leq T$$

## Study of three algorithms: How to model $\nabla_x \log p_t(v | x_t)$ ?

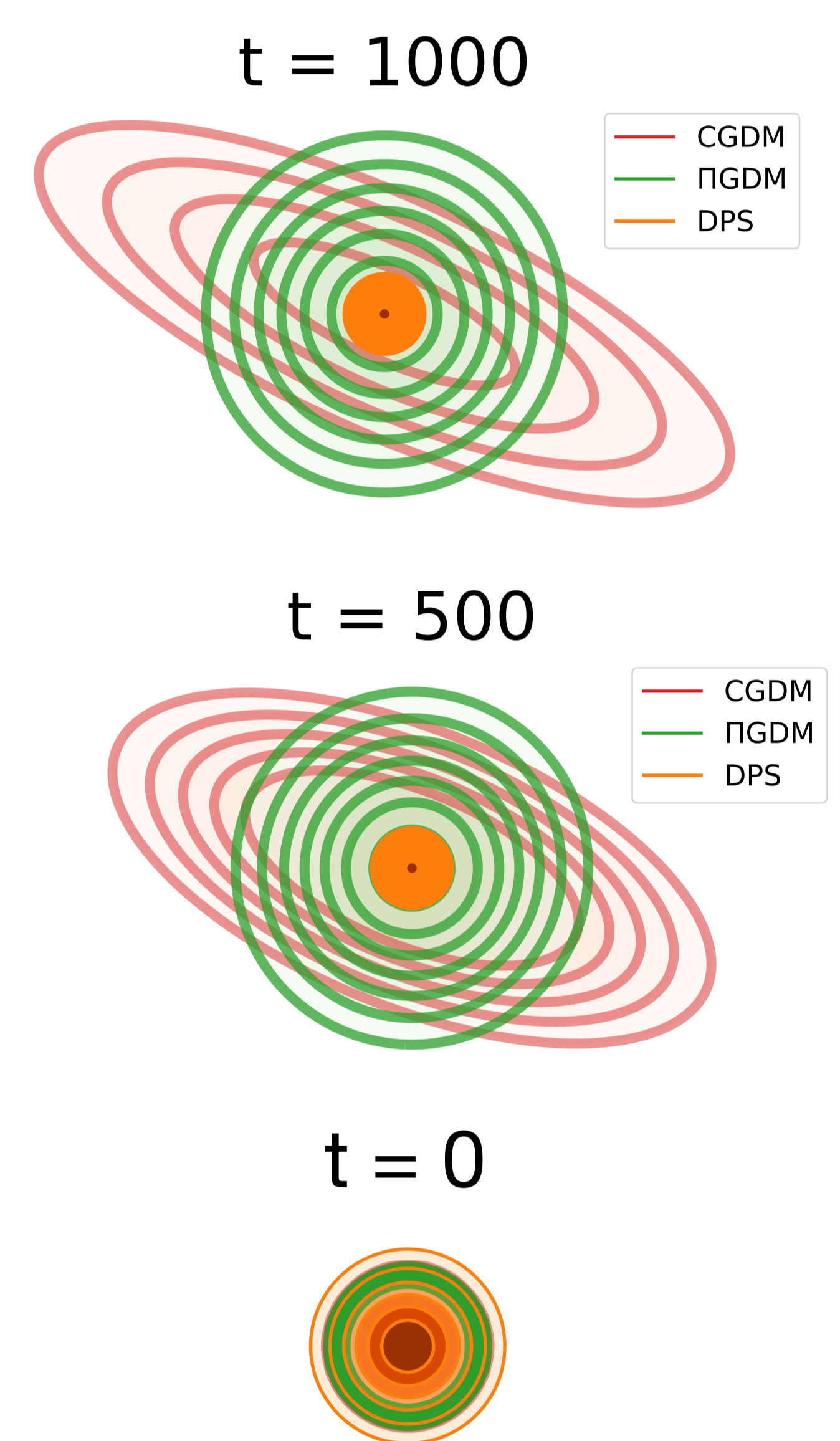
$\hat{x}_0(x_t) := \mathbb{E}(x_0 | x_t)$  is given by Tweedie's formula and

$$\mathbb{E}(v | x_t) = A \hat{x}_0(x_t) = \frac{1}{\sqrt{\bar{\alpha}_t}} A (x_t + (1 - \bar{\alpha}_t) \nabla_x \log p_t(x_t)) \quad \text{with } \bar{\alpha}_t = \prod_{s=1}^t (1 - \beta_s)$$

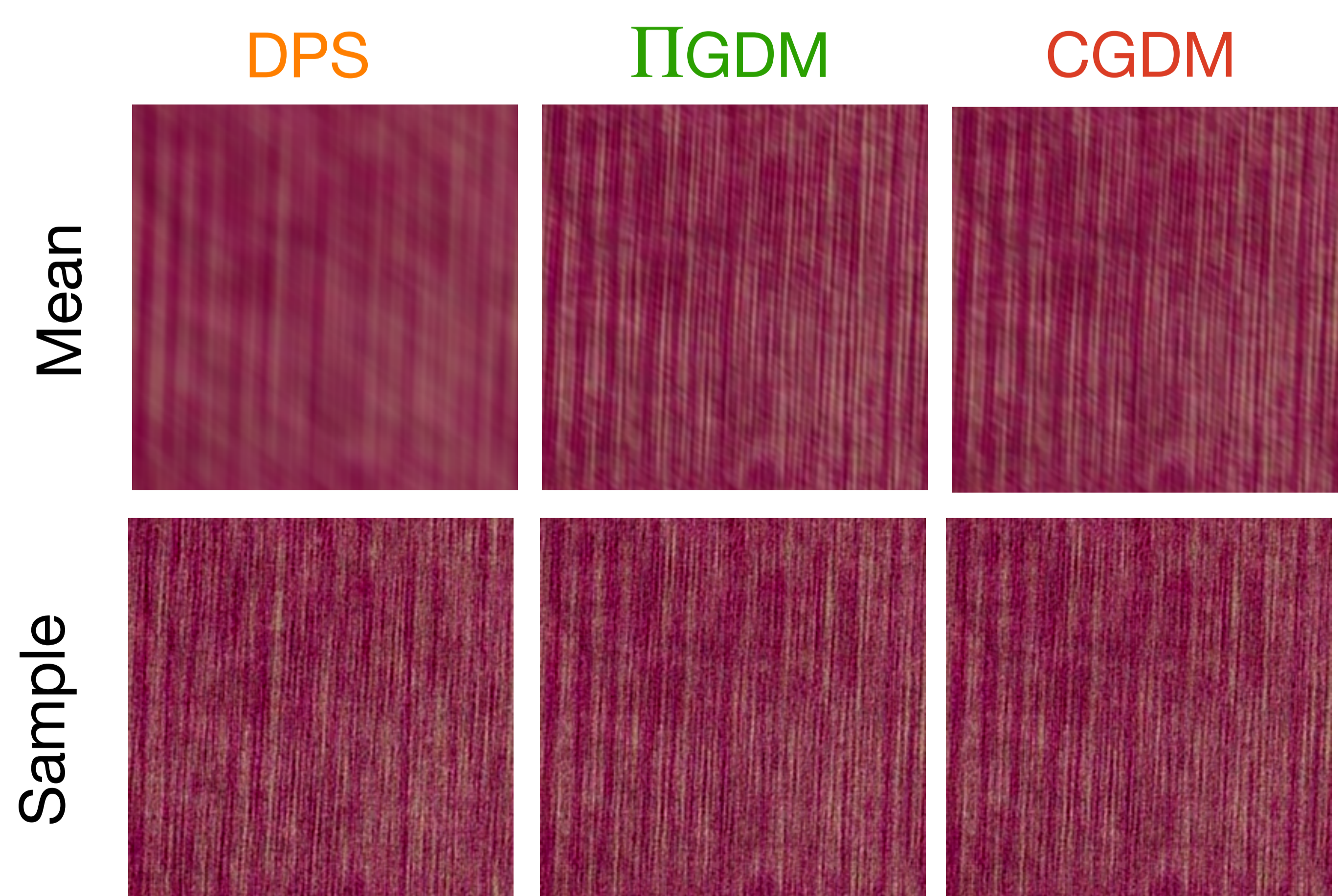
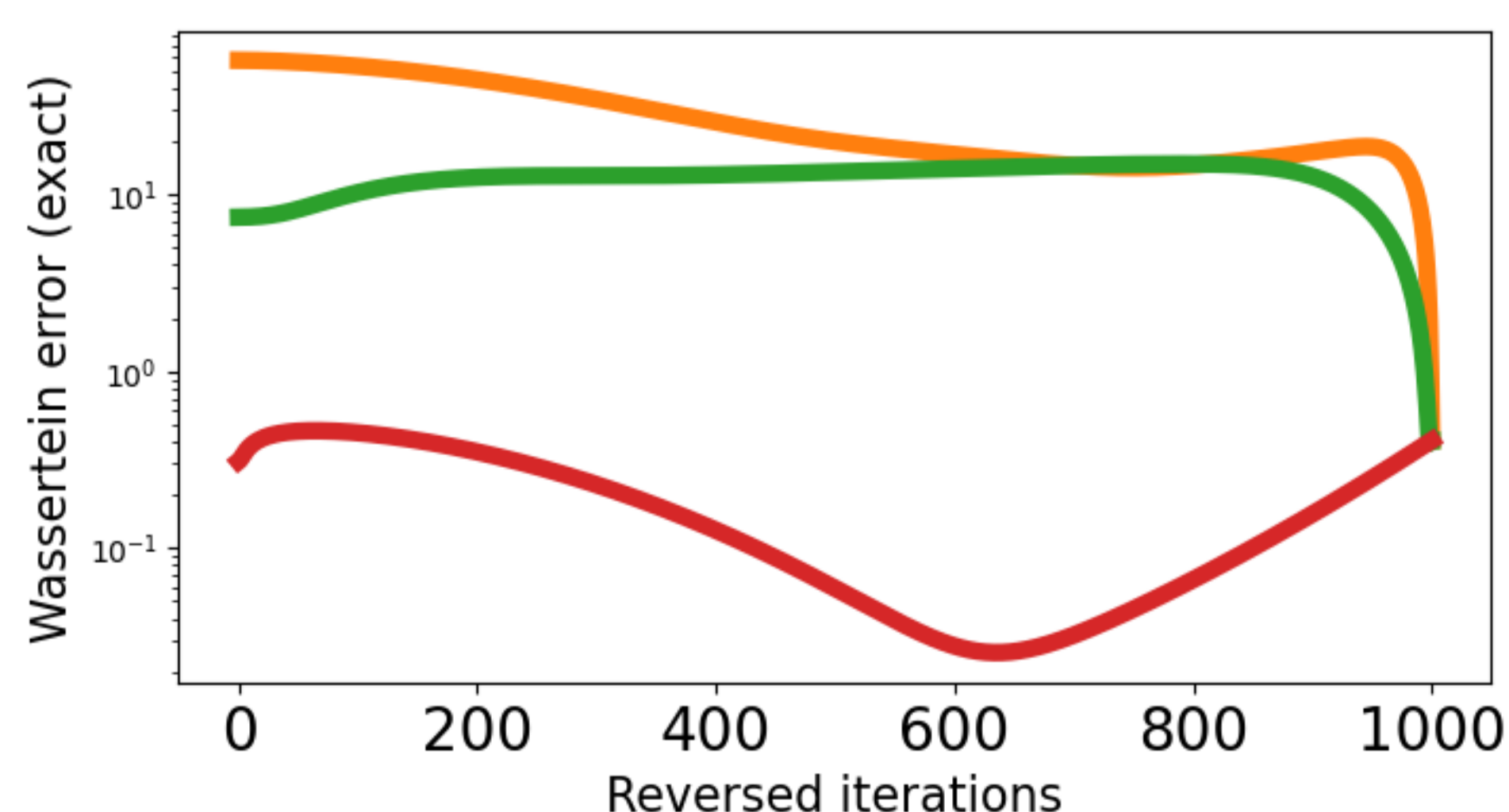
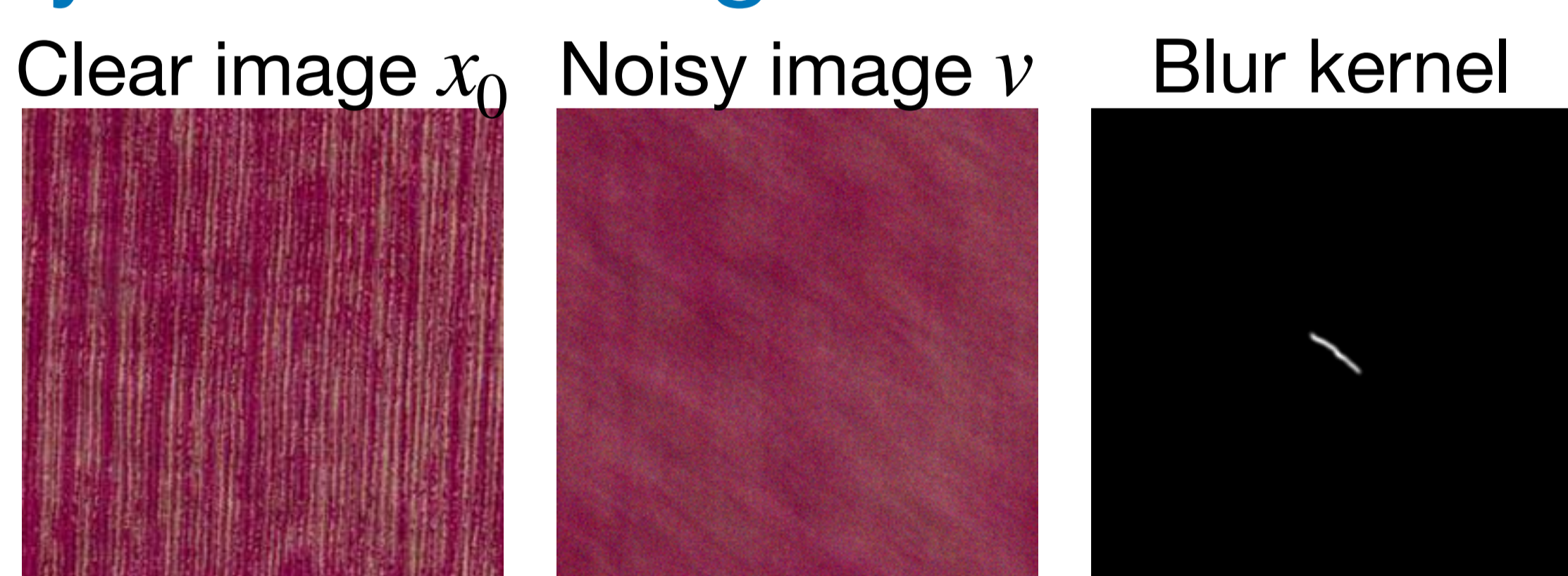
The distribution  $p(v | x_t)$  is modelled by a Gaussian  $\mathcal{N}(A \hat{x}_0(x_t), C_{v|t})$  such that

$$\nabla_x \log p(v | x_t) = -\frac{1}{2} \nabla_x \|v - A \hat{x}_0(x_t)\|_{C_{v|t}^{-1}}^2$$

Algorithms	$C_{v t}$
DPS [1]	$\frac{\sigma^2}{\alpha_{\text{DPS}}} \mathbf{I}$
$\Pi$ GDM [2]	$(1 - \bar{\alpha}_t) \mathbf{A} \mathbf{A}^T + \sigma^2 \mathbf{I}$
CGDM (Gaussian case)	$(1 - \bar{\alpha}_t) \mathbf{A} \mathbf{\Sigma} \mathbf{\Sigma}^{-1} \mathbf{A}^T + \sigma^2 \mathbf{I}, \quad \mathbf{\Sigma}_t = \bar{\alpha}_t \mathbf{\Sigma} + (1 - \bar{\alpha}_t) \mathbf{I}$



## Study of deblurring of Gaussian textures



## Conclusion

- In terms of Wasserstein error, **CGDM** < **DPS** <  **$\Pi$ GDM**.
- **DPS** produces biased samples and instabilities.
- In terms of computation time, **DPS** <  **$\Pi$ GDM** < **CGDM**.

## References

- [1] Chung, H., Kim, J., Mccann, M.T., Klasky, M.L., Ye, J.C.: Diffusion posterior sampling for general noisy inverse problems. ICLR (2023)
- [2] Song, J., Vahdat, A., Mardani, M., Kautz, J.: Pseudoinverse-guided diffusion models for inverse problems. ICLR (2023)



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